CURRICULUM, PEDAGOGY AND BEYOND

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Integrating Eigenvalues and Eigenvectors into Linear Transformation Education

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Dr Robin's Python Lab

When Maths Meets Python: 100 Coding Experiments for Problem Solvers

Robin Wang

Apple Book

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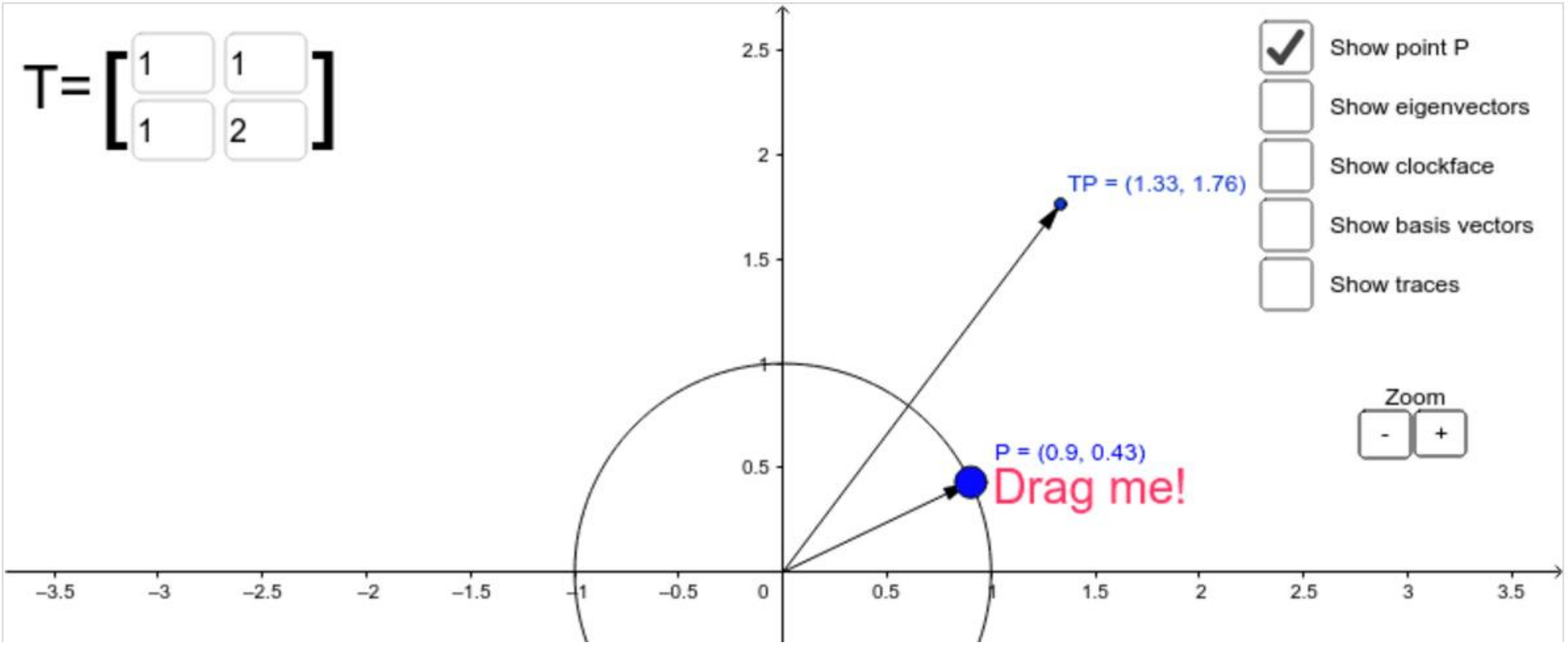
Mobi

Linear transformations by matrices

$$T(x) = Ax$$

- For most vectors, the transformation changes both their direction and magnitude. However, eigenvectors are unique in that they only change in magnitude (scaled), not in direction.



Mathematical definitions of eigenvalues and eigenvectors

- Given a square matrix A and a vector v, an eigenvector v and its corresponding eigenvalue \lambda satisfy the equation:

$$Av = \lambda v$$

- This means when the matrix A acts on the vector v, the result is scaled version of v, with the scale factor being \lambda.

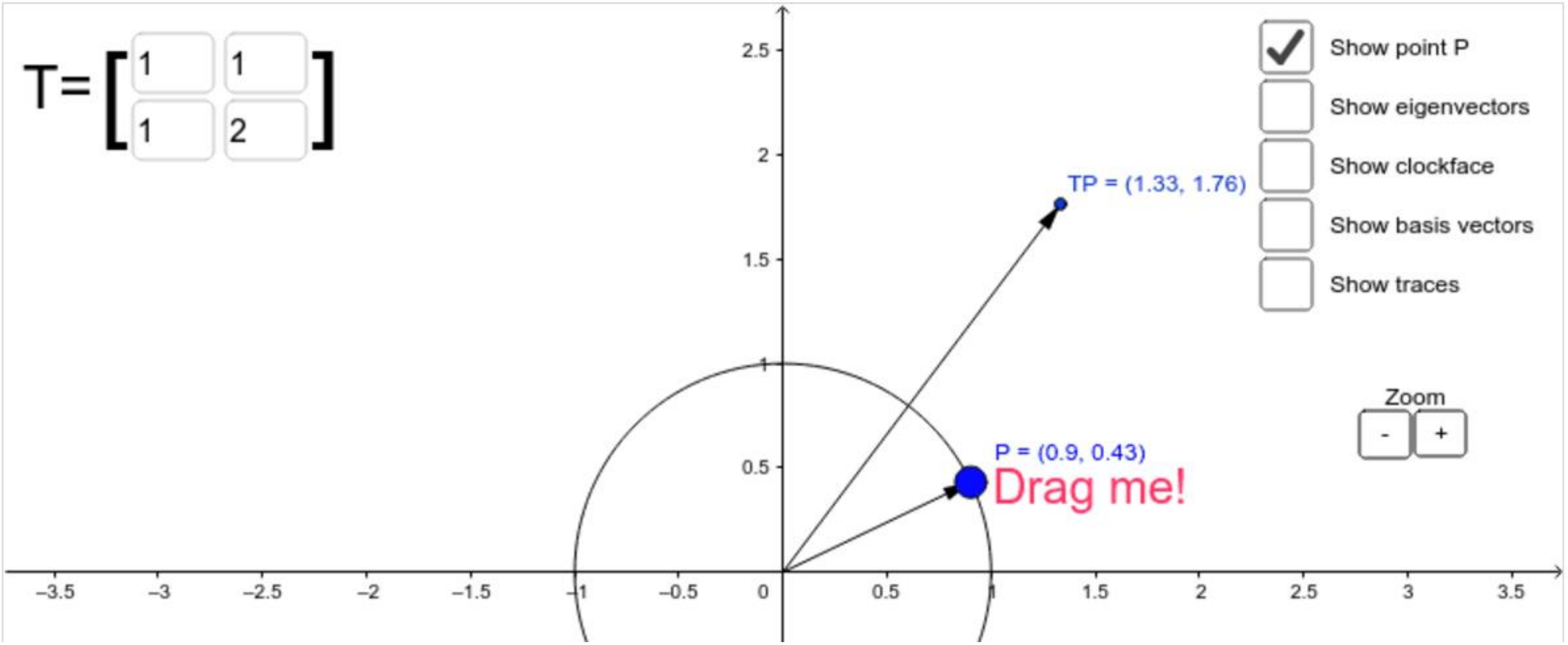
Interpreting eigenvalues and eigenvectors

- Eigenvalues are the factors by which the eigenvectors are scaled during the transformation. An eigenvalue tells you how much the eigenvector gets stretched (if $|\lambda| > 1$) or shrunk (if $0 < |\lambda| < 1$) after the transformation.

- Analogy: You could think of an eigenvector as a “stick” lying along the direction of the transformation. The eigenvalue would represent how much the stick gets stretched or compressed during the transformation.

Visualising eigenvectors and eigenvalues

- For example, a 2 by 2 matrix can be visualized as a transformation of a 2D space, and the eigenvectors can be shown as those special directions where the vectors simply stretch or compress, not rotate.



Real-world applications

- Principal component analysis (PCA): In Data Science, PCA uses eigenvectors to identify the “directions” in which the data varies most, reducing dimensionality.

- Stability analysis: In Control Theory, eigenvalues help in determining the stability of equilibrium points.

A simple example

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$$

$$\lambda = 3,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t,\ t \in R$$

$$\lambda = 7,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t,\ t \in R$$

Understand linear transformations

- A diagonal matrix represents a linear transformation where the basis vectors are scaled by the diagonal values. Diagonalising a matrix helps us understand how a transformation acts on the space more clearly by breaking it down into simpler, independent scaling operations along each axis defined by eigenvectors.

When can you diagonalize a matrix?

- Not all matrices can be diagonalized. A matrix can be diagonalized if and only if it has a full set of linearly independent eigenvectors.

- A key result in linear algebra, known as the Spectral Theorem, states that for any real symmetric matrix A, there exists an orthogonal matrix P (where P^{-1} = P^T) and a diagonal matrix D, such that:

$$P^{-1}AP = D \Rightarrow P^{T}AP = D$$

- Where the diagonal elements of D are the eigenvalues of A, and the columns of P are the orthonormal eigenvectors of A.

Conic sections and their symmetric matrices representations

- Conic sections – ellipse, parabola, and hyperbola – can be represented by quadratic equations in two variables. These equations are typically written as:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Simplification of conic sections via diagonalization

- To simplify conic sections using diagonalization, the goal is to eliminate the mixed product term bxy and reduce the quadratic equation to a form that is easier to work with.
- Matrix representation: The quadratic form of the conic section can be written in a symmetric matrix format. This allows us to apply matrix diagonalization techniques.

$$x=\begin{bmatrix} x \\ y \end{bmatrix},\ A=\begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow x^T A x = ax^2 + 2bxy + cy^2$$

$$v=\begin{bmatrix} d \\ e \end{bmatrix} \Rightarrow x^T A x + v^T x + f = ax^2 + 2bxy + cy^2 + dx + ey + f$$

Conic section in symmetric matrix format

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0 \Rightarrow x^T A x + v^T x = -f$$

$$x = Px',\ x' = \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad |P| = 1 \qquad P^T A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- The eigenvectors form the axes of the new coordinate system (rotated axes), where the conic section is represented without the cross term bxy. The eigenvalues determine the dilation factor along these axes.

Ellipse, hyperbola or parabola

$$I_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

- If $I_2 > 0$, the conic is an ellipse.
- If $I_2 < 0$, it represents a hyperbola.
- If $I_2 = 0$, it corresponds to a parabola (or two parallel lines).

Example 1 (IB Cambridge)

$$5x^2 + 4xy + 5y^2 = 21 \Rightarrow x^T A x = 21 \textit{ where } A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & -2 \\ -2 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 7)$$

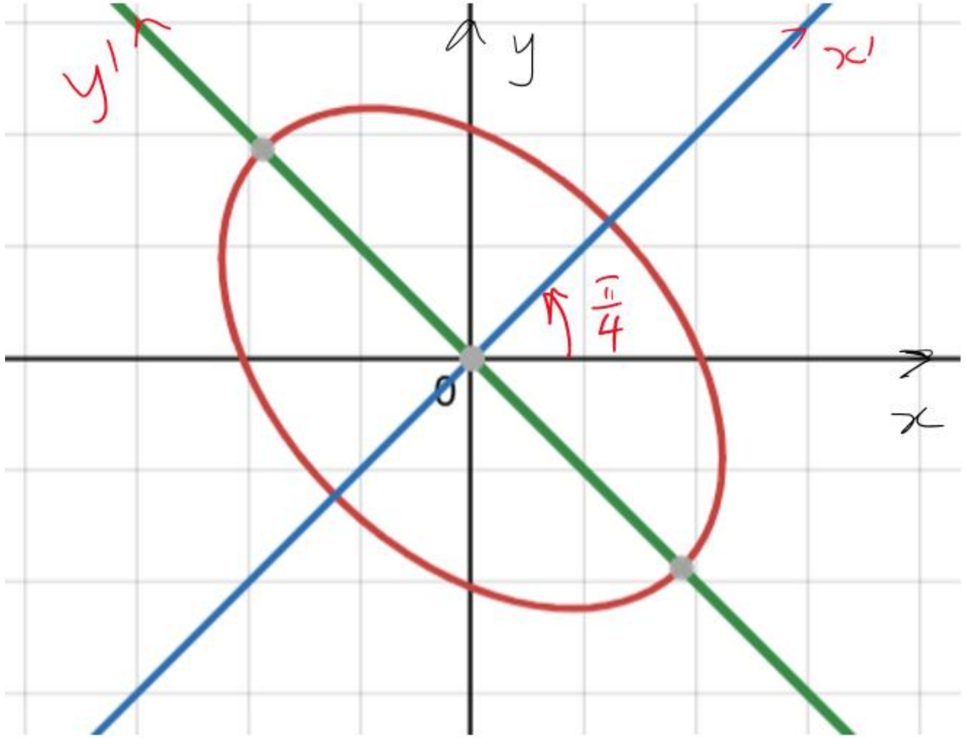
$$\lambda = 3,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t,\ t \in R \qquad \lambda = 7,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t,\ t \in R$$

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Example 1, cont.

$$x = Px' \Rightarrow (Px')^T A (Px') = 21 \Rightarrow x'^T P^T A P x' = 21 \Rightarrow x'^T \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} x' = 21$$

$$\Rightarrow 7x'^2 + 3y'^2 = 21 \Rightarrow \frac{x'^2}{3} + \frac{y'^2}{7} = 1$$



Example 2 (IB Cambridge)

$$2x^2 - 4xy - y^2 = -8$$

$$2x^2 - 4xy - y^2 = -8 \Rightarrow x^{\mathrm{T}}Ax = -8 \; where \; A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix} = -8$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 2 \\ 2 & \lambda + 1 \end{vmatrix} = (\lambda - 3)(\lambda + 2)$$

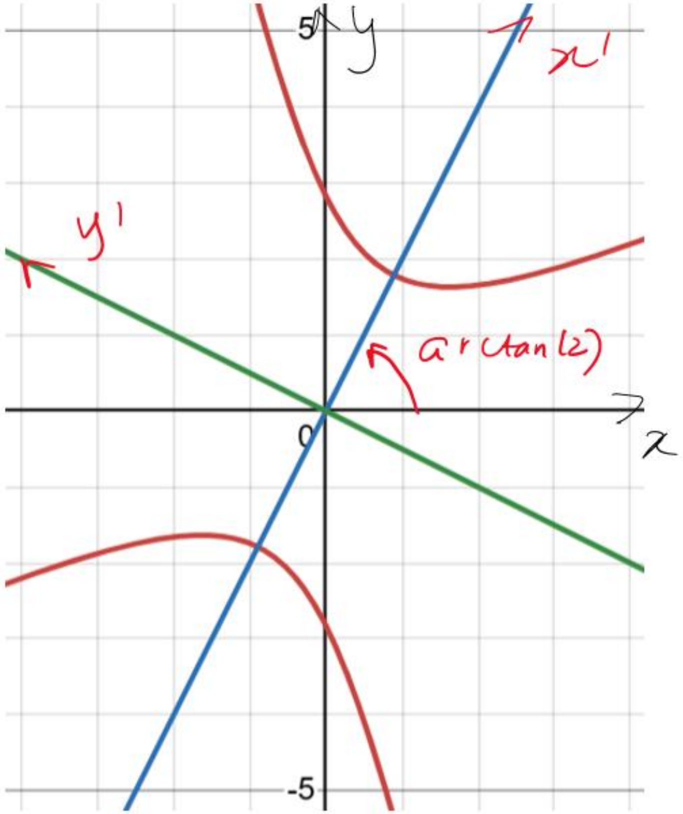
$$\lambda = 3,\; (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t,\; t \in R \qquad \lambda = -2,\; (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t,\; t \in R$$

$$P = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Example 2, cont.

$$x = Px' \Rightarrow (Px')^T A (Px') = -8 \Rightarrow x'^T P^T A P x' = -8 \Rightarrow x'^T \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} x' = -8$$

$$\Rightarrow 2x'^2 - 3y'^2 = 8 \Rightarrow \frac{x'^2}{4} - \frac{y'^2}{\frac{8}{3}} = 1$$



Example 3 (IB Cambridge)

$$24x^2 - 14xy - 24y^2 - 60x + 80y = -100$$

$$24x^2 - 14xy - 24y^2 - 60x + 80y = -100 \Rightarrow \mathbf{x}^{\mathrm{T}} A\mathbf{x} + \mathbf{v}^{\mathrm{T}}\mathbf{x} = -100 \ \textit{where}\ A = \begin{bmatrix} 24 & -7 \\ -7 & -24 \end{bmatrix}, v = \begin{bmatrix} -60 \\ 80 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 24 & 7 \\ 7 & \lambda + 24 \end{vmatrix} = (\lambda + 25)(\lambda - 25)$$

$$\lambda = 25,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} -7 \\ 1 \end{bmatrix} t,\ t \in R \qquad \lambda = -25,\ (\lambda I - A)x = 0 \Rightarrow x = \begin{bmatrix} 1 \\ 7 \end{bmatrix} t,\ t \in R$$

$$P = \frac{1}{\sqrt{50}} \begin{bmatrix} 1 & -7 \\ 7 & 1 \end{bmatrix}$$

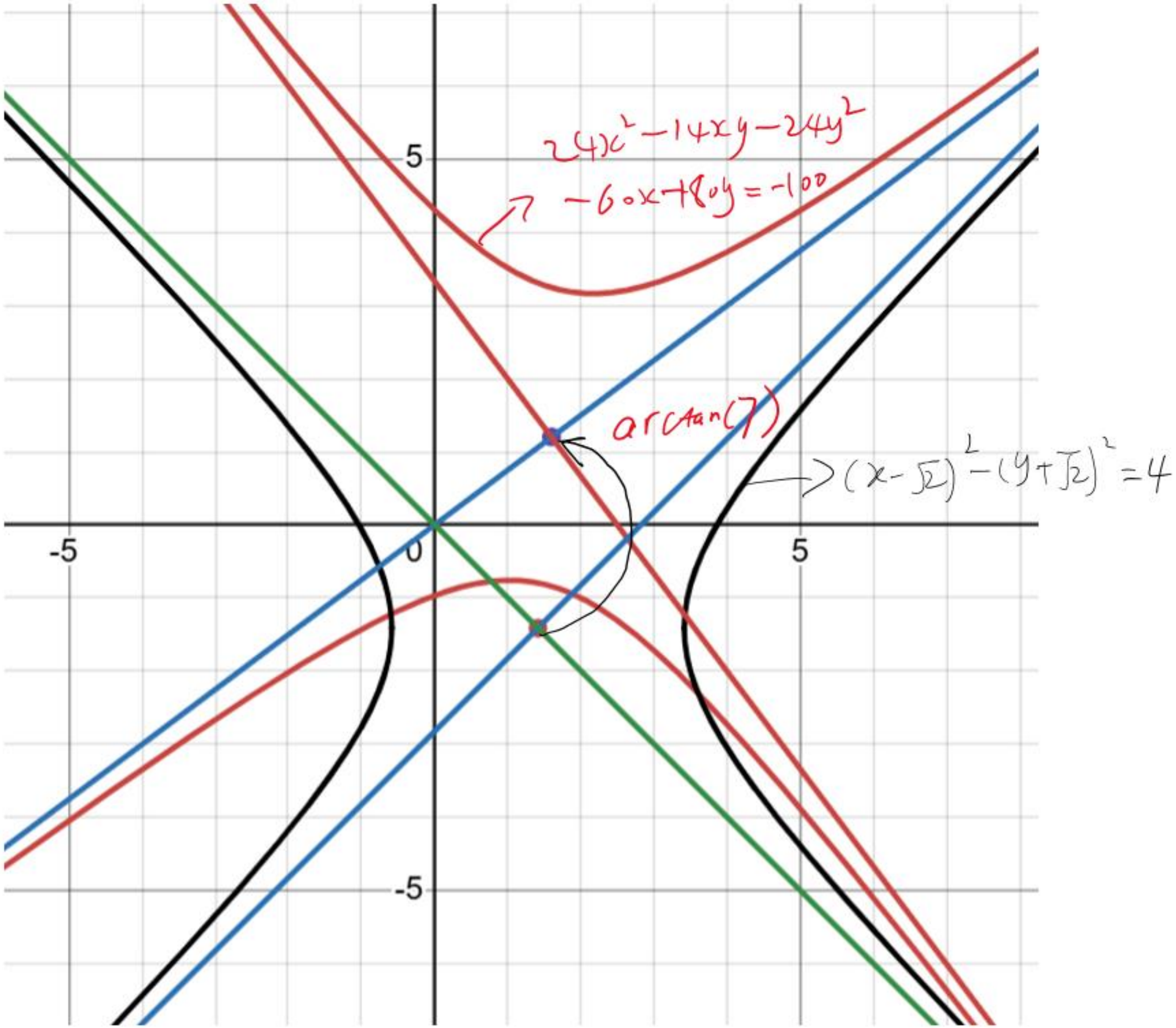
Example 3, cont.

$$x = Px' \Rightarrow (Px')^T A (Px') + v^T P x' = -100 \Rightarrow x'^T P^T A P x' + \begin{bmatrix} -60 & 80 \end{bmatrix} \frac{1}{\sqrt{50}} \begin{bmatrix} 1 & -7 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = -100$$

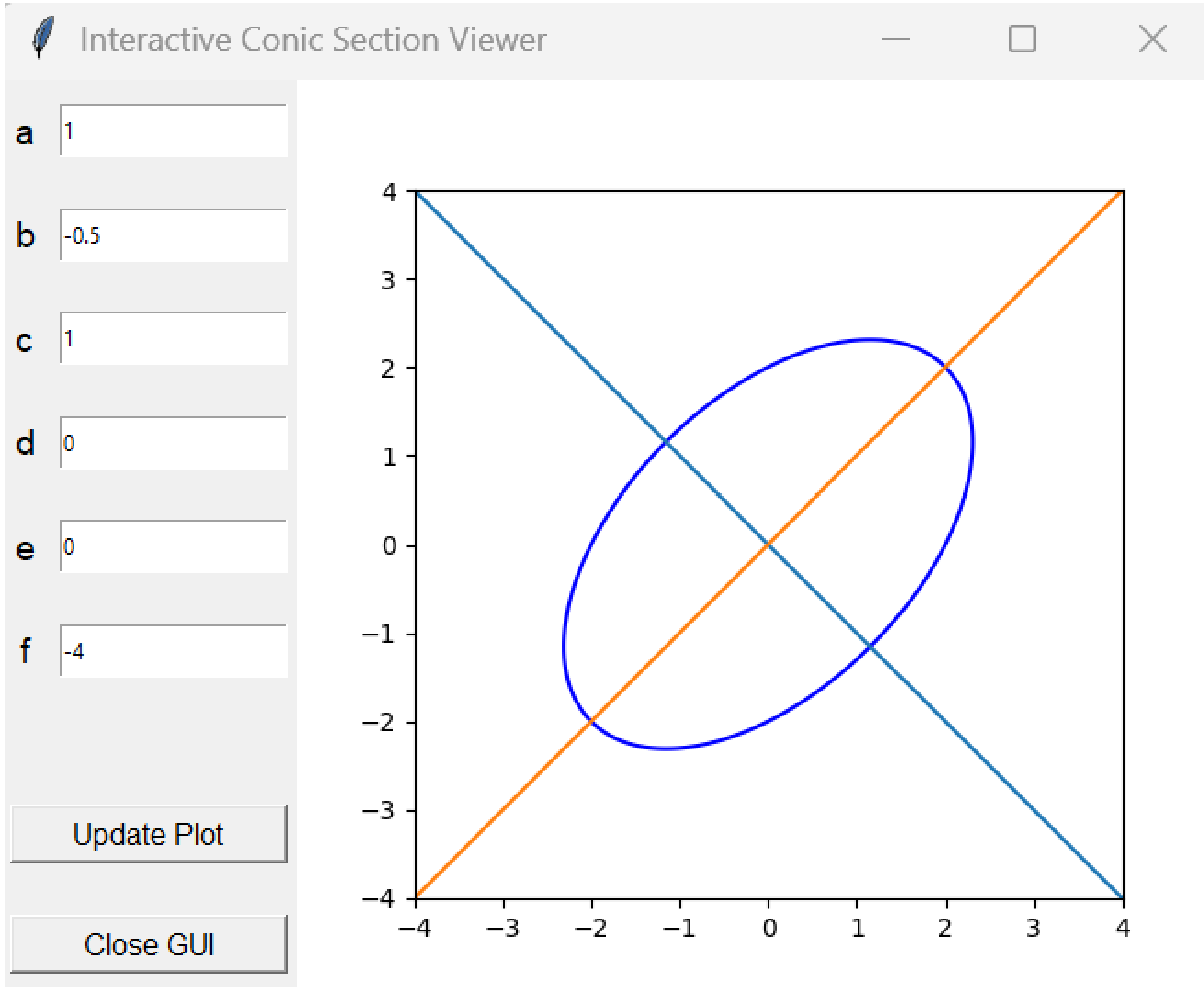
$$\Rightarrow x'^T \begin{bmatrix} -25 & 0 \\ 0 & 25 \end{bmatrix} x' + \frac{1}{5\sqrt{2}} \begin{bmatrix} 500 & 500 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = -100$$

$$\Rightarrow x'^2 - y'^2 - 2\sqrt{2}x' - 2\sqrt{2}y' = 4$$

$$\Rightarrow \left(x' - \sqrt{2}\right)^2 - \left(y' + \sqrt{2}\right)^2 = 4$$


24x² − 14xy − 24y²
− 60x + 80y = −100
arctan(7)
(x − √2)² − (y + √2)² = 4
5
-5
0
5
-5

Interact with Python



Questions & Discussions?

Event App

App Download Instructions

Step 1: Download the App 'Arinex One' from the App Store or Google Play

App Store

Google Play

Step 2: Enter Event Code: **mav**

Step 3: Enter the email you registered with

Step 4: Enter the Passcode you receive via email and click 'Verify'. Please be sure to check your Junk Mail for the email, or see the Registration Desk if you require further assistance.

Be in it to WIN!